

Thermal gravity, black holes and cosmological entropy

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Taking seriously the interpretation of black hole entropy as the logarithm of the number of microstates, we argue that thermal gravitons may undergo a phase transition to a kind of black hole condensate. The phase transition proceeds via nucleation of black holes at a rate governed by a saddlepoint configuration whose free energy is of order the inverse temperature in Planck units. Whether the universe remains in a low entropy state as opposed to the high entropy black hole condensate depends sensitively on its thermal history. Our results may clarify an old observation of Penrose regarding the very low entropy state of the universe.

I. INTRODUCTION

Years ago Penrose noticed that the universe must have begun in a very low entropy state [1]. By considering the entropy of black holes, he argued that the current state of the universe has significantly lower entropy than the maximum possible entropy state. For example, while holding the number of baryons fixed one could increase the total entropy tremendously by letting matter collapse into black holes [2]. Indeed, it seems that while the matter degrees of freedom were born hot, i.e., in a maximum entropy thermal state, the gravitational degrees of freedom were born in a very special low entropy state. Interpreting entropy as the logarithm of phase space volume, a low entropy state is an exponentially unlikely state and hence can only result from fine-tuned initial conditions [3]. Reasoning along these lines suggests that spacetimes with numerous horizons, perhaps resembling a dense agglomeration of black holes, occupy an exponentially larger fraction of gravitational phase space than smooth spacetimes like the usual Friedmann-Robertson-Walker (FRW) cosmologies. For related discussions, see, e.g., [4, 5] and references contained therein.

One may ask whether special initial conditions at the Planck scale are sufficient to produce the low entropy universe we see today. It might be the case that interactions with thermal matter in the early universe inevitably cause the gravitational degrees of freedom to thermalize as well. Such a thermal state, assuming ergodicity of gravity, would likely evolve to a configuration of much higher entropy, and hence a cosmology very different from the one we observe.

Since black holes are our only hint at the highly entropic configurations of gravity [8], they should play a prominent role in the transition from low entropy to high entropy spacetimes. In this paper we suggest a specific mechanism involving the nucleation of black holes from a thermal graviton state. We note that the corresponding nucleation rate from a thermal matter state is much smaller, and probably irrelevant cosmologically.

The mechanism we describe provides a plausible means by which Penrose's ergodic evolution could proceed. We examine whether the transition to a new, highly entropic, phase of condensed black holes can occur in standard big bang cosmology. The result depends sensitively on the thermal history of the universe at early times. Moreover, the relevant energy scales are all higher than the energy scale at which an inflationary epoch is usually assumed to take place. Therefore, we are considering a phase transition which only may take place before and not after inflation. Presumably, the probability for a given patch to inflate would be affected by whether or not that patch has undergone a phase transition to the high entropy phase.

We should note that there is not a consensus on the issue of whether gravity is ergodic nor on the interpretation of the gap between the maximum allowed and the actual entropy in an FRW spacetime. Tipler [6] showed that under a reasonable set of assumptions, closed universes are technically not ergodic, i.e., there is no Poincaré recurrence. Moreover, Barrow [7] has pointed out that in a spacetime restricted to be FRW there is necessarily an entropy gap, i.e., the entropy in thermal radiation is much less than the entropy associated with a black hole of horizon size. However, the phenomena we described in the previous paragraph, which are investigated in this paper, are independent of these larger questions about general relativity. That is, the mechanism by which black holes are nucleated occurs on sub-horizon time and length scales. The statistical approach we take below is justified by the presence of a thermal bath of gravitons or other particles, whose existence is not in dispute. In this sense we do not require any assumption of ergodicity, *except in some small sub-horizon patch*.

In Sec. II we consider black hole nucleation in a system of thermal gravitons and compare to a thermal system of matter. In Sec. III we show that gravitons may thermalize in the early universe even if they started out cold. We determine the conditions necessary for a phase transition to a black hole condensate via percolation in Sec. IV. Finally, in Sec. V we relate these results to Penrose's observation. We use Planck units throughout, i.e., $\hbar = c = G = k_B = 1$.

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II. STATISTICAL MECHANICS OF GRAVITONS

Consider a box of hot gravitons. The probability for a fluctuation to lead to a black hole of radius R is

$$P(R) \sim Ne^{-E/T}, \quad (1)$$

where $E = R/2$ is the energy of the black hole, and N is the multiplicity of microstates which, when coarse grained, appear as a black hole of radius R . The probability can be written as

$$P(R) \sim e^{-F/T}, \quad (2)$$

where $F = E - TS$ is the free energy, and $S = \ln N$. We assume that black hole entropy is accounted for by gravitational microstates, as suggested by results from string theory [11]. Using the Bekenstein-Hawking formula [12] for black hole entropy $S_{\text{BH}} = A/4$, where $A = 4\pi R^2$ is the area, we see that

$$F(R) = R/2 - \pi T R^2. \quad (3)$$

Strictly speaking, we want the free energy *relative* to that of hot, flat space. This means we should subtract from (3) a correction $F_0(R) \sim -R^3 T^4$. It is easy to see that, near the saddlepoint found below, F_0 is a negligible correction as long as the saddlepoint radius R_* is much smaller than the horizon size.

The radius that maximizes the free energy, the saddlepoint radius, is given by $R_* = (4\pi T)^{-1}$. We obtain $1 \ll R_* \ll H^{-1}$, where $H^{-1} \sim 1/T^2$ is the horizon size for a radiation dominated FRW universe, so our analysis encounters no difficulties from quantum gravity or causality. The corresponding maximum free energy is (see Fig. 1)

$$F_* = (16\pi T)^{-1}. \quad (4)$$

At the saddlepoint radius the black hole temperature is just equal to that of the heat bath. Black holes with $0 < R < R_*$ shrink to zero size, leaving a weak-field phase with a thermal population of graviton states (gravity waves) on a smooth background metric. However, for $R > R_*$, black holes grow without bound (they are colder than the environment), and for $R > R_0 = (2\pi T)^{-1}$ the free energy is actually negative, less than that of $R = 0$. This instability may indicate a new nonperturbative phase of gravity, which is not asymptotically flat, and in which spacetime is all or partially filled with black holes. Such a phase is highly entropic and occupies an exponentially larger phase space volume than the smooth weak-field phase.

The nucleation rate for supercritical black holes (which might be thought of as bubbles of the new nonperturbative gravitational phase) is controlled by the free energy at the saddlepoint, as in the usual case of a first order phase transition (for early papers on nucleation theory, see [13]). This yields

$$\lambda(T) \sim T^4 e^{-F_*/T}. \quad (5)$$

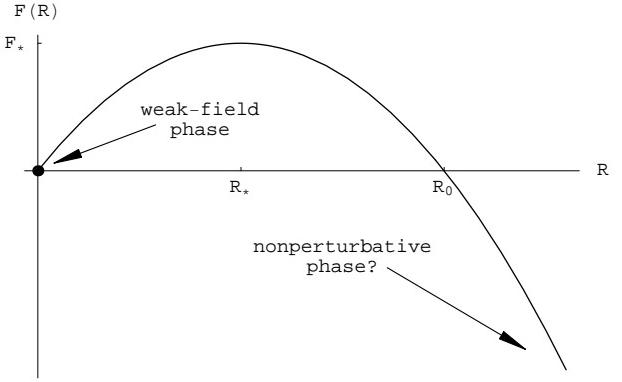


FIG. 1: Free energy versus black hole radius for a gas of hot gravitons. Black holes of size $R < R_*$ shrink, leading to a weak-field gravity phase. For $R > R_*$, however, black holes grow, possibly giving rise to a new nonperturbative phase of gravity.

(Strictly speaking, the dimensional prefactor could be modified by subleading terms in the exponent.) The physics is similar to that of nonperturbative baryon number violation in the standard model. There, the rate is controlled by the free energy of the electroweak sphaleron, which is the saddlepoint configuration separating vacua of different winding number [14].

Now consider a box of hot photons. If a fluctuation of size R and energy E satisfies $E > R$ (we ignore factors of order one), it will inevitably evolve into a black hole [9]. Therefore, Eqns. (1) and (2) for the probability for a fluctuation to lead to a black hole of radius R still hold. In this case, however, the entropy is not proportional to A . In order to evaluate the multiplicity N , we make use of a bound on the entropy of a region of size R filled with thermal radiation, originally derived by 't Hooft [10]. By noting that matter in thermal equilibrium, energy scales as $E \sim R^3 T^4$, and entropy as $S \sim R^3 T^3$, and further requiring that the system not have already undergone gravitational collapse, i.e., $E < R$, 't Hooft obtained the bound $S < A^{3/4}$ (again, we ignore numerical factors). Matter configurations that lead to black holes saturate this bound. Therefore, we see that the free energy of relativistic matter configurations which evolve into black holes is of the form $F(R) \sim R - TR^{3/2}$.

Once a fluctuation of sufficient size to lead to a black hole has occurred, the evolution is then governed by the same physics as in the original graviton case. We therefore compare the multiplicities of configurations of photons and gravitons, N_γ and N_g , respectively, that will lead to black holes of critical size R_* . For temperatures well below the Planck scale, $A \gg A^{3/4}$, and so fluctuations of critical size in the photon gas are suppressed relative to the case of a graviton gas by a factor of roughly

$$\frac{N_\gamma}{N_g} \sim \exp\left(-\frac{1}{16\pi T^2}\right). \quad (6)$$

The key difference between the two cases considered is

the entropy limit on thermal degrees of freedom. Ordinary matter cannot achieve the entropic density of gravitational degrees of freedom, under the assumption that black holes are coarse grained objects with $e^{A/4}$ gravitational microstates. Nucleation of black holes is much more likely if these gravitational microstates are thermally occupied (i.e., in a graviton heat bath), than if one starts with hot matter and cold gravitons.

III. GRAVITON THERMALIZATION

Natural initial conditions for the universe might have both gravitons and matter in thermal equilibrium. In some cases, however, thermal matter leads to thermal graviton populations even when the gravitons are initially cold. We can investigate the thermalization of gravitons using a long-wavelength effective field theory (EFT) for quantum gravity [15, 16]. If one is only interested in processes occurring at energies sufficiently below the Planck scale, as is the case here, there is no obstacle to using the standard EFT approach of including all terms in the effective lagrangian that are consistent with the symmetries of the system, in this case general coordinate transformation invariance.

Without knowledge of the fundamental theory of quantum gravity, we are not able to write down the Boltzmann equation for the evolution of the phase space distribution for the gravitational degrees of freedom. Boltzmann heuristics, however, motivate using $\Gamma(T) = H(T)$ as a reasonable criterion for freeze-out of a given particle species, in this case the graviton. $\Gamma(T)$ is the interaction rate of gravitons with the heat bath at temperature T , and $H(T)$ is the Hubble expansion rate at temperature T . Below the decoupling temperature T_{dec} the Hubble expansion rate is greater than the interaction rate of gravitons with the heat bath, and gravitons are decoupled.

Let us estimate the graviton decoupling temperature based on two different scattering processes. First, consider the interaction $X_i X_i \rightarrow gg$, where X_i is any one of N scalar, fermion or vector particles, and g is the graviton. In a relativistic gas, Boltzmann heuristics suggest that the interaction rate is roughly $\Gamma_{X_i X_i \rightarrow gg} \sim n_i \sigma_i$, where n_i and σ_i are the number density and cross section, respectively, of species i . At energies not too far below the Planck scale all particle species are relativistic so $n_i \sim T^3$. The matrix element for this process goes as T^2 and the Hubble rate as $H \sim N^{1/2}T^2$. We see that the decoupling temperature obtained from considering the process $\sum_i X_i X_i \rightarrow gg$ is $T_{\text{dec}} \sim N^{-3/2}$. For a model with a large number of particle species this scale may be well below the Planck scale.

This estimate, however, does not take into account interactions between particle species. A similar argument based on considering the process $X_i X_i \rightarrow \gamma g$, where γ is another particle species, leads to a decoupling temperature of roughly $T_{\text{dec}}^\gamma \sim N^{-1/2}\alpha^{-1}$, where $\alpha^{1/2}$ is the

coupling between X_i and γ . Again T_{dec}^γ may be significantly below the Planck scale.

Let us obtain a more careful estimate of T_{dec}^γ in a specific model: long-wavelength quantum gravity (EFT) coupled to scalar QED (see, e.g., [17]). For a tree level calculation, the relevant terms in the effective lagrangian are:

$$\begin{aligned} \mathcal{L} = & |g|^{\frac{1}{2}} [g^{\mu\nu}(D_\mu\phi)^*(D_\nu\phi) - \frac{1}{4}g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}] \\ & + \frac{1}{16\pi}|g|^{\frac{1}{2}}\mathcal{R}, \end{aligned} \quad (7)$$

where $D_\mu = \partial_\mu + ieA_\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and \mathcal{R} is the scalar curvature. The metric $g_{\mu\nu}$ is expanded about the Minkowski metric $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, i.e., $g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{32\pi}h_{\mu\nu}$, and Feynman rules are obtained by the standard procedure. The cross section for the interaction $\phi\phi \rightarrow \gamma g$ is found to be

$$\sigma_{\phi\phi \rightarrow \gamma g} = \frac{32\pi}{6}\alpha, \quad (8)$$

where $\alpha = e^2/4\pi$. Assuming local thermal equilibrium, the scalar number density is $n_\phi = (2\zeta(3)/\pi^2)T^3$. Moreover, if there are N_s species of scalar particles, the graviton decoupling temperature is

$$T_{\text{dec}}^\gamma \simeq \frac{3}{5\alpha N_s^{1/2}}. \quad (9)$$

For consistency we must also estimate the temperature below which we can trust the above calculation, i.e., estimate the energy scale below which the EFT description is valid. Let us look at the next order contribution to the matrix element for this process:

$$\mathcal{M}_{\phi\phi \rightarrow \gamma g} = \mathcal{M}_{\phi\phi \rightarrow \gamma g}^{(1)} + \mathcal{M}_{\phi\phi \rightarrow \gamma g}^{(3)} + \dots \quad (10)$$

Notice $\mathcal{M}_{\phi\phi \rightarrow \gamma g}^{(1)} = \mathcal{O}(e\sqrt{32\pi}q)$, where q is some typical energy scale of the interaction. In order to obtain a conservative estimate of $\mathcal{M}_{\phi\phi \rightarrow \gamma g}^{(3)}$, we will assume that all diagrams interfere constructively. Then $\mathcal{M}_{\phi\phi \rightarrow \gamma g}^{(3)} = \mathcal{O}(FLe(\sqrt{32\pi}q)^3)$, where F is the number of graphs at this order and L is a loop factor associated with each graph. The EFT is valid for

$$q \ll q_{\text{pert}} = (32\pi FL)^{-1/2}. \quad (11)$$

Taking $F = 27$ and $L = 1/4\pi^2$ we get $q_{\text{pert}} \simeq 10^{-1}$. By comparing Eqs. (9) and (11), we see that $T_{\text{dec}}^\gamma \ll q_{\text{pert}}$ in the large N_s limit, due to the fact that $T_{\text{dec}}^\gamma \sim N_s^{-1/2}$, while q_{pert} is independent of N_s . Thus, there is a class of models (those with large numbers of matter fields) in which gravitons are copiously produced and interact frequently, at energy scales where the effective field theory applies.

One could perform a similar calculation with a more realistic model for the matter content of the early universe. At the energy scales of interest, say 10^{16} GeV, the

matter interactions may, for example, be described by a grand unified theory (GUT) [18] with $N_s \sim 10^2$ and couplings not much smaller than unity. A calculation in such a model would likely yield the same conclusion, i.e., that there was an epoch in which even initially cold gravitons interacted strongly enough with the heat bath that they were themselves thermalized.

IV. PERCOLATION OF BLACK HOLES

In light of the previous sections, it is interesting to consider the possibility of a first order phase transition in an FRW universe with a thermal population of gravitons. Assuming that gravitons are in thermal equilibrium in the early universe between an initial time t_0 and a later time t_1 , the volume fraction in the weak-field gravity phase is given by:

$$p(t_0, t_1) = \exp \left[- \int_{t_0}^{t_1} dt' V(t', t_1) \lambda(t') \right]. \quad (12)$$

In flat FRW spacetime

$$V(t', t_1) = \frac{4\pi}{3} \left[a(t') b \int_{t'}^{t_1} \frac{dt''}{a(t'')} \right]^3. \quad (13)$$

Here a is the scale factor, and b is the (constant) speed at which the black holes expand. Eq. (12) is a standard formula from old inflationary cosmology [19]. b is governed by relativistic physics and, therefore, should not be significantly smaller than unity. During a radiation dominated epoch the scale factor goes as $a(t) \sim t^{1/2}$ and temperature and time are related by $t \simeq 0.3g_*^{-1/2}T^{-2}$. The volume fraction (now as a function of temperature) is then of the form:

$$p(T_0, T_1) = \exp \left[- \frac{b^3}{g_*^2} f(T_0, T_1) \right]. \quad (14)$$

We define the critical initial temperature T_0^c , so that if the initial graviton temperature is larger than T_0^c , the volume fraction in the weak-field phase is nearly zero. See Fig. 2 for a plot of $p(T_0, T_1)$ for $b = 10^{-1}, 10^{-2}$, and 10^{-3} , with $T_1 = 10^{-6}$ and $g_* = 10^2$. Decreasing the expansion speed by two orders of magnitude has the effect of increasing T_0^c by less than a factor of two. Increasing the number of effective degrees of freedom or the cut-off temperature T_1 both have similar effects on T_0^c , i.e., they cause it to increase by a relatively small amount.

The critical initial temperature T_0^c is of order 10^{-2} to 10^{-1} , which from Eq. (11) is the same order of magnitude as the temperature at which quantum gravity becomes perturbative. Thus, taking the initial temperature of the big bang to be roughly the scale at which gravity becomes perturbative, the universe could avoid a phase transition to the nonperturbative black hole phase, assuming it begins in the weak-field phase.

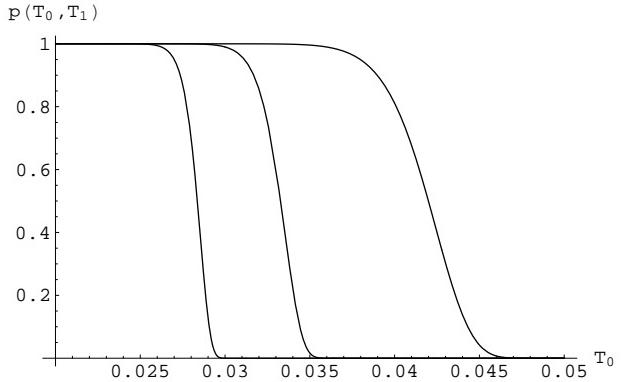


FIG. 2: Volume fraction in the weak-field phase of gravity as a function of the temperature T_0 at which gravitons enter thermal equilibrium. Shown are plots for three different values of the expansion speed, $b = 10^{-1}, 10^{-2}, 10^{-3}$ (left to right). Here the cut-off temperature $T_1 = 10^{-6}$, and the effective number of degrees of freedom $g_* = 10^2$.

V. DISCUSSION

We examined a possible first order phase transition of spacetime to a black hole phase with high entropy. Percolation of the high entropy phase occurs if gravitons are ever in a thermal state with temperature above T_0^c , either because they were born hot at the Planck epoch or because they were thermalized due to interactions with thermal matter. It seems possible, as suggested by entropic arguments, that almost all of gravitational phase space is accounted for by the nonperturbative phase. However, we find that the low entropy phase is metastable over timescales which are exponentially sensitive to the temperature, and potentially quite long.

One may wonder how inflation changes this conclusion. We note that T_0^c is higher than the energy scale at which inflation is usually assumed to take place. If gravitons are never thermalized above a temperature of T_0^c then presumably inflation would simply take place as originally envisioned. However, if gravitons are ever thermal with a temperature above T_0^c then we would speculate that it may be less probable for a given patch to inflate, although depending on the details of the model some non-zero probability may remain, even if a phase transition to the high entropy black hole phase does occur.

Hot gravitons with temperature slightly below T_0^c will not lead to a phase transition; they will simply be redshifted away. Both gravitons and matter may be born hot, as long as the temperature of the universe (either initially or after a period of inflation) is never greater than T_0^c . This does not require fine tuning because T_0^c is of the same order as q_{pert} , the energy scale below which quantum gravity effects are small. It may still be the case that the initial conditions represent a subset of measure zero in the total phase space, which is dominated by the nonperturbative black hole phase [20]. However, our

analysis does show that once the initial choice of the low entropy phase is made, no transition to the high entropy phase need occur.

These conclusions remain unchanged in a spacetime of arbitrary dimension d . For hot gravitons in d dimensions, the exponent governing the nucleation rate of Eq. (5) goes as $F_*/T \sim T^{-(d-2)}$, while for matter $F_*/T \sim T^{-(d-1)(d-2)/2}$. For $d > 3$, black hole nucleation is suppressed more strongly in the matter system than in the gravitational one. Moreover, as $d \rightarrow \infty$, T_0^c increases, meaning a transition to the black hole condensate phase is less likely in a universe with a large number of spacetime dimensions.

Note added: After this paper was completed we became aware of earlier work using Euclidean path inte-

gral methods in which Eqs. (3), (4), and (5) were independently derived [21]. In these calculations imaginary time boundary conditions are applied to the gravitational field. Therefore, those authors were also studying thermal gravity and not only thermal matter. For discussion of black hole phase transitions, see [22].

Acknowledgments

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